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A distributive semilattice not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group ☆

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Abstract

We construct a distributive 0-semilattice which is not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group. The size of the semilattice is $\aleph_2$.

Introduction

The notion of ideal appears in many algebraic theories. All ideals of a given algebraic object form typically an algebraic lattice, usually modular, in many interesting cases distributive. It is natural to ask which algebraic lattices can be represented as ideal lattices of algebraic objects of a certain type.

It is well known that two-sided ideals of a von Neumann regular ring form an algebraic distributive lattice. Thus, one of the representation problems can be formulated as follows: which algebraic distributive lattices are isomorphic to the lattices of two-sided ideals of von Neumann regular rings, in particular, as the lattices of two-sided ideals of locally matricial algebras (i.e., direct limits of finite products of full matrix rings $\mathbb{M}_n(k)$ over a field k).

G.M. Bergman [1] constructed a locally matricial algebra with the ideal lattice isomorphic to an algebraic lattice L provided that either the lattice L is strongly distributive or the set of its compact elements is at most countable. The author [7] proved that every

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algebraic distributive lattice whose semilattice of compact elements is closed under finite meets is isomorphic to the lattice of two-sided ideals of some locally matricial algebra. On the negative side, F. Wehrung [10] constructed an algebraic distributive lattice which is not isomorphic to the lattice of two-sided ideals of any von Neumann regular ring. The cardinality of the set of compact elements of his counterexample is $\aleph_2$. The same author [11] proved that every algebraic distributive lattice with at most $\aleph_1$ compact elements is isomorphic to the ideal lattice of some von Neumann regular ring. However, the ring is not a locally matricial algebra (it is not even a unit regular ring) and thus the following problem remains open.

Problem 1. Let L be an algebraic distributive lattice with at most $\aleph_1$ compact elements. Does there exist a locally matricial algebra with the lattice of two-sided ideals isomorphic to L? (Compare to [5, Problem 10.2] and [11, Problem 1].)

The lattice of two-sided ideals of a locally matricial algebra R is isomorphic to the lattice of ideals (i.e., directed convex subgroups) of its Grothendieck group $\mathsf{K}_0(R)$. As locally matricial algebras are direct limits of directed families of matricial algebras, the corresponding Grothendieck groups are direct limits of directed families of simplicial groups. The direct limits of directed families of simplicial groups were characterized by E.G. Effros, D.E. Handelman, and C.-L. Shen [2] to be unperforated, directed partially ordered Abelian groups satisfying the interpolation property. These groups are called dimension groups.

K.R. Goodearl and D.E. Handelman [4] proved that any dimension group of size at most $\aleph_1$ is isomorphic to the Grothendieck group of some locally matricial algebra while F. Wehrung [9] constructed a dimension group of size $\aleph_2$ which is not isomorphic to the Grothendieck group of any von Neumann regular ring.

In view of the result of K.R. Goodearl and D.E. Handelman, Problem 1 can be restated in terms of dimension groups and their ideal lattices.

Problem 1′. Let L be an algebraic distributive lattice with at most $\aleph_1$ compact elements. Does there exist a dimension group G with the lattice of ideals isomorphic to L? (See [5, Problem 10.1′].)

To prove that a given algebraic distributive lattice L occurs as the lattice of ideals of a dimension group, it suffices to show that the semilattice of compact elements of L occurs as the semilattice of finitely generated ideals of the group. The semilattice of finitely generated ideals of a dimension group is isomorphic to the maximal semilattice quotient of its positive cone.

In the paper, we construct a distributive 0-semilattice T, of size $\aleph_2$, which is not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group. We have not succeeded to modify the construction to get a counterexample of smaller cardinality, and so Problem 1′ remains open. Nevertheless, we answer negatively a more general problem formulated in [5].

Problem 10.1 [5] (*Lifting distributive semilattices to dimension groups*). Let S be a distributive 0-semilattice. Does there exist a dimension group G such that the maximal semilattice quotient of G^+ is isomorphic to S?

1. Basic concepts

Lattices, semilattices

All semilattices considered in the text are join semilattices.

Let L be a complete lattice and let a be an element of L. Then a is called *compact*, if for any $A \subseteq L$, $a \leqslant \bigvee A$ implies that $a \leqslant \bigvee F$ for some finite $F \subseteq A$. The set of compact elements of a complete lattice is closed under finite joins, and thus forms a 0-semilattice. Note that this set is not closed under finite meets in general.

A complete lattice L is called *algebraic* if every element of L is a join of compact elements.

A nonempty lower subset of a semilattice S closed under finite joins is called an *ideal of* S. A lattice L is algebraic if and only if it is isomorphic to the lattice of all ideals of some 0-semilattice [6, II.3. Theorem 13], in particular, if it is isomorphic to the lattice of all ideals of the semilattice of its compact elements.

Let Θ be a congruence relation of a semilattice S. For $x \in S$, we denote by $[x]\Theta$ the congruence class containing x. Given an ideal I of a semilattice S, we denote by $\Theta[I]$ the smallest congruence relation Θ of S under which $x \equiv y(\Theta)$ for all $x, y \in I$. Note that for every $u, v \in S$, $u \equiv v(\Theta[I])$ if and only if $u \vee x = v \vee x$ for some $x \in I$.

A semilattice S is called *distributive* if for every $a, b, c \in S$ such that $c \leqslant a \vee b$ there exist $a', b' \in S$ with $a' \leqslant a$, $b' \leqslant b$, and $c = a' \vee b'$. A semilattice S is distributive if and only if its ideal lattice is distributive [6, II.5. Lemma 1].

The *refinement property* is the following semigroup-theoretical axiom: A commutative semigroup C satisfies the *refinement property* if and only if $a_0 + a_1 = b_0 + b_1$, $(a_0, a_1, b_0, b_1 \in C)$ implies the existence of $c_{ij} \in C$ $(i, j = 0, 1)$ with $a_i = c_{i0} + c_{i1}$ $(i = 0, 1)$ and $b_j = c_{0j} + c_{1j}$ $(j = 0, 1)$. A commutative monoid is a *refinement monoid* if it satisfies the refinement property. By induction we get immediately

Lemma 1.1. *Let C be a commutative semigroup satisfying the refinement property. If $x = x_i^0 + x_i^1$ $(x, x_i^0, x_i^1 \in G^+)$ for all $i < n$, then there exists a set $\{x_f \mid f \in 2^n\}$ of elements of G^+ such that*

$$x_i^0 = \sum_{f(i)=0} x_f \quad \textit{and} \quad x_i^1 = \sum_{f(i)=1} x_f$$

for every $i < n$.

Note that semilattices are exactly idempotent commutative semigroups. On the other hand, for a commutative semigroup C there exists a least congruence $\asymp$ on C such that $C/{\asymp}$ is a semilattice. The congruence $\asymp$ is defined as follows: for every $a, b \in C$, $a \asymp b$ if and only if there exist a positive integer n such that $a \leqslant nb$ and $b \leqslant na$. The quotient $C/\asymp$ is denoted by $\nabla(C)$ and called the *maximal semilattice quotient of* C.

It is an easy fact that a semilattice is distributive if and only if it satisfies the refinement property. Consequently, if a commutative semigroup C satisfies the refinement property, then $\nabla(C)$ is a distributive semilattice.

The correspondence that with a commutative monoid M associate $\nabla(M)$ naturally extends to a direct limits preserving functor from the category of commutative monoids to the category of semilattices (see [5, Section 2]). For $x \in M$ we denote by $[x]$ the corresponding element of $\nabla(M)$.

A nonzero element a of a semilattice S is *join-irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$ for every $b, c \in S$. We denote by $\mathsf{J}(S)$ the partially ordered set of all join-irreducible elements of a semilattice S. A distributive semilattice in which every element is a finite join of join-irreducible elements is called *strongly distributive.* The easy proof of the following lemma is left to the reader.

Lemma 1.2. *Let S be the* 0*-semilattice of compact elements of an algebraic lattice L. Then the following conditions are equivalent*:

(i) *The semilattice S is strongly distributive.*
(ii) *The lattice L is isomorphic to the lattice of lower subsets of the partially ordered set* $\mathsf{J}(S)$.
(iii) *The lattice L is isomorphic to the lattice of lower subsets of some partially ordered set.*

Dimension groups

The *positive cone* of a partially ordered Abelian group G is a monoid $G^+ = \{a \in G \mid 0 \leqslant a\}$, and we put $G^{++} = \{a \in G \mid 0 < a\}$. If $f : G \to H$ is an order preserving homomorphism of partially ordered Abelian groups G, H, then $f[G^+] \subseteq H^+$. We denote by f^+ the restriction of f from G^+ to H^+.

A partially ordered Abelian group G is called *unperforated* if $na \geqslant 0$ implies $a \geqslant 0$ for all $a \in G$ and every positive integer n.

A partially ordered Abelian group G is called *directed* if every element of G is a difference of two elements of G^+. Note that a partially ordered Abelian group is directed if and only if it is directed as a partially ordered set.

A partially ordered set P satisfies the *interpolation property* if for every a_0, a_1, b_0, and b_1 in P such that $a_i \leqslant b_j$ $(i, j = 0, 1)$, there exists $c \in P$ such that $a_i \leqslant c \leqslant b_j$ $(i, j = 0, 1)$. An *interpolation group* is a partially ordered Abelian group which satisfies the interpolation property. A partially ordered Abelian group is an interpolation group if and only if its positive cone is a refinement monoid [3, Proposition 2.1].

A *dimension group* is an unperforated, directed, interpolation group. E.G. Effros, D.E. Handelman, and C.-L. Shen characterized dimension groups as direct limits of directed diagrams of simplicial groups [2, Theorem 2.2], where a *simplicial group* is a partially ordered Abelian group isomorphic to $\mathbb{Z}^n$ with the positive cone $(\mathbb{Z}^+)^n$ for some positive integer n.

A directed convex subgroup of a partially ordered Abelian group is called an *ideal*. We denote by $\mathsf{Id}(G)$ the lattice of all ideals of a partially ordered Abelian group G, and by $\mathsf{Id}_c(G)$ the semilattice of compact elements of the lattice $\mathsf{Id}(G)$. It is easily seen that the semilattice $\mathsf{Id}_c(G)$ is isomorphic to the maximal semilattice quotient $\nabla(G^+)$ of the positive cone of a partially ordered Abelian group G.

An ordered vector space is a partially ordered Abelian group endowed with a structure of vector space over the totally ordered field $\mathbb{Q}$ of rational numbers such that the multiplication by a positive scalar is order-preserving. An ordered vector space which is directed and satisfies the interpolation property is called a *dimension vector space*. Note that every ordered vector space is unperforated. With a partially ordered Abelian group G we associate the ordered vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$ with the positive cone $G^+ \otimes_{\mathbb{Z}} \mathbb{Q}^+$. The correspondence $[a] \mapsto [a \otimes 1]$ defines an isomorphism $\nabla(G^+) \to \nabla(G^+ \otimes_{\mathbb{Z}} \mathbb{Q}^+)$, and so every distributive 0-semilattice isomorphic to the maximal semilattice quotient of the positive cone of some dimension group is, at the same time, isomorphic to the maximal semilattice quotient of some dimension vector space.

2. Weakly Archimedean partially ordered Abelian groups

Definition 2.1. A partially ordered Abelian group G is called *weakly Archimedean* if for every $a, b \in G^+$, there exists a positive integer n such that $na \nleqslant b$.

Definition 2.2. Let G be a partially ordered Abelian group, let $V = G \otimes_{\mathbb{Z}} \mathbb{Q}$ be the ordered vector space associated to G. For $a, b \in G^+$ we define

$$(a/b) = \sup\left\{q \in \mathbb{Q}^+ \mid a \otimes 1 \geqslant b \otimes q \text{ in } V\right\}.$$

Note that a partially ordered Abelian group G is weakly Archimedean if and only if for every $a, b \in G^{++}$, $(a/b) < \infty$. We state some simple properties of the map $(-/-) : G^+ \times G^+ \to [0, \infty]$.

Lemma 2.3. *Let G, H be partially ordered Abelian groups. Let a, a', b, and b' be elements of G^+. Then*

(i) $(a/b) > 0$ *if and only if* $[a] \geqslant [b]$;
(ii) *if* $[a] = [a']$ *and* $[b] = [b']$, *then* $(a/b) = \infty$ *if and only if* $(a'/b') = \infty$;
(iii) *if* $f : G \to H$ *is an order preserving homomorphism, then* $(f(a)/f(b)) \geqslant (a/b)$.

Let G be a partially ordered Abelian group. For $a, b \in G^+$ we use the notation $a \perp b$ if $c \leqslant a, b$ implies that $c = 0$ for every $c \in G^+$.

Lemma 2.4. *Let G be an interpolation group. If $a \perp b$ $(a, b \in G^+)$, then $a, b \leqslant c$ implies that $a + b \leqslant c$ for every $c \in G^+$.*

Proof. Since G is an interpolation group, there exists $d \in G$ such that $a, b \leqslant d \leqslant a + b, c$. Then $0 \leqslant a + b - d \leqslant a, b$, and since $a \perp b$, we get that $a + b = d \leqslant c$. □

Corollary 2.5. *Let G be an interpolation group. Let $a, b, c \in G^+$ satisfy $a \perp b$ and $a \leqslant c$. Then $(c/b) = (c - a/b)$.*

Lemma 2.6. *Let c, d, b be elements of the positive cone of an ordered vector space V. If $(c + d/b) = \infty$ and $(b/d) > 0$, then $(c/d) = \infty$.*

Proof. From $(b/d) > 0$, it follows that there exists a positive integer m such that $d \leqslant mb$. Thus, for all positive integers n, $m(n+1)b \leqslant c + d \leqslant m(c+b)$, whence $(n+1)b \leqslant c + b$. Therefore, $nb \leqslant c$. $\square$

Lemma 2.7. *Let P be a partially ordered upwards directed set. Let*

$$\langle G, f_p : G_p \to G \rangle_{p \in P} = \varinjlim \langle G_p, f_{p,q} : G_p \to G_q \rangle_{p \leqslant q \text{ in } P}$$

in the category of partially ordered Abelian groups. Let $p \in P$. Then, for every $a, b \in G_p^+$

(i) *if $p \leqslant q$ in P, then $(a/b) \leqslant (f_{p,q}(a)/f_{p,q}(b))$;*
(ii) $(f_p(a)/f_p(b)) = \sup\{(f_{p,q}(a)/f_{p,q}(b)) \mid p \leqslant q \text{ in } P\}$.

Moreover, if P is σ-directed (*i.e., every countable subset of P is bounded*)*, then there exists q in P above p, such that*

$$\big(f_p(a)/f_p(b)\big) = \big(f_{p,q}(a)/f_{p,q}(b)\big).$$

In particular, if $(f_p(a)/f_p(b)) = \infty$, then there exists q in P above p, with $(f_{p,q}(a)/f_{p,q}(b)) = \infty$.

3. Example

In this section, we construct in two steps a distributive 0-semilattice which is not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group. First we define a strongly distributive 0-semilattice S which is not isomorphic to the maximal semilattice quotient of the positive cone of any weakly Archimedean dimension group. The cardinality of the semilattice S is $\aleph_2$. Then, by means of the semilattice S, we construct a distributive 0-semilattice T, of the same cardinality $\aleph_2$, which is not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group.

In connection with the first step of our construction, note that from the result of G.M. Bergman [1], it follows that every strongly distributive 0-semilattice is isomorphic to the maximal semilattice quotient of the positive cone of some dimension group. K.R. Goodearl and F. Wehrung defined for every distributive lattice D a dimension vector space $\mathbb{Q}\langle D\rangle$ such that $\nabla(\mathbb{Q}\langle D\rangle^+)$ is isomorphic to D as a join semilattice [5, Section 4]. It is readily seen that the dimension vector space $\mathbb{Q}\langle D\rangle$ is weakly Archimedean. Further, they proved that every countable distributive 0-semilattice is isomorphic to the maximal semilattice quotient of the positive cone of some countable dimension vector space

[5, Theorem 5.2]. We begin with proving that also in this case the dimension vector space can be taken weakly Archimedean.

Theorem 3.1. *Every countable distributive* 0*-semilattice is isomorphic to the maximal semilattice quotient of the positive cone of some countable weakly Archimedean dimension vector space.*

Proof. Let S be a countable distributive 0-semilattice. Let $\sigma : S \to \nabla(\mathbb{Q}^+)$ be the $\langle\vee, 0\rangle$-homomorphism defined by $\sigma(0) = [0]$ while $\sigma(x) = [1]$ for $x > 0$. By [8, Theorem 7.1], there are a countable dimension vector space G, a positive homomorphism $f : G \to \mathbb{Q}$, and an isomorphism $\varepsilon : S \to \nabla(G^+)$ such that $\sigma = \nabla(f^+) \circ \varepsilon$. For every $a, b \in G^{++}$, $0 < f(a), f(b)$, and, by Lemma 2.3(iii), $(a/b) \leqslant (f(a)/f(b)) = f(a)/f(b)$. It follows that G is weakly Archimedean. □

Theorem 3.2. *There exists a strongly distributive semilattice S, of size $\aleph_2$, which is not isomorphic to the maximal semilattice quotient of any weakly Archimedean dimension group.*

Proof. We define S to be a strongly distributive 0-semilattice with

$$\mathsf{J}(S) = \{z\} \cup \{y_\beta \mid \beta < \omega_1\} \cup \{x_\alpha \mid \alpha < \omega_2\}$$

ordered by the relations $z > y_\beta > x_\alpha$ $(\alpha < \omega_2, \beta < \omega_1)$.

To the contrary, suppose that there exists a weakly Archimedean dimension group G and an isomorphism $\phi : \nabla(G^+) \to S$. Without loss of generality, we can assume that G is a dimension vector space. Pick c, b_β $(\beta < \omega_1)$, and a_α $(\alpha < \omega_2)$ from G^+ such that $(\phi[c]) = z$, $\phi([b_\beta]) = y_\beta$ for every $\beta < \omega_1$, and $\phi([a_\alpha]) = x_\alpha$ for every $\alpha < \omega_2$. In addition, we can choose b_β so that $c > b_\beta$, thus, $(c/b_\beta) \geqslant 1$ for every $\beta < \omega_1$.

For all $\alpha < \omega_2$, $\beta < \omega_1$, put $\mu_{\alpha,\beta} = (b_\beta/a_\alpha)/3$. Note that $0 < \mu_{\alpha,\beta} < \infty$ for every $\alpha < \omega_2$, $\beta < \omega_1$. Let Ω be the family of all nonempty finite subsets of ω_1. Choose a map $F : \omega_2 \to \Omega$ so that

$$\forall \alpha < \omega_2\colon \sum_{\beta \in F(\alpha)} \mu_{\alpha,\beta} > (c/a_\alpha). \tag{3.1}$$

Observe that $\mathsf{card}(F^{-1}(\{B\})) = \aleph_2$ for some $B = \{\beta_0, \ldots, \beta_{n-1}\} \in \Omega$. For each $i < n$, put $b_i^1 = b_{\beta_i}$ and $b_i^0 = c - b_{\beta_i}$. Since $c > b_{\beta_i}$, $b_i^0 \in G^+$ for every $i < n$, and so, by Lemma 1.1, there exist b_f $(f \in 2^n)$ in G^+ such that

$$b_i^0 = \sum_{f(i)=0} b_f \quad \text{and} \quad b_i^1 = \sum_{f(i)=1} b_f \quad (i < n).$$

Let χ_i $(i < n)$ denote the characteristic function of $\{i\}$. Further, denote by D the set of maps $f : n \to 2$ that satisfy $f(i) = 1 = f(j)$ for at least two different non-negative integers i, j

smaller than n. If $f \in D$, then $[b_f]$ lies below at least two elements of the set $\{[b_{\beta_i}] \mid i < n\}$, hence there exists a finite subset A_f of ω_2 such that

$$[b_f] = \bigvee_{\alpha \in A_f} [a_\alpha].$$

Put $A = \bigcup_{f \in D} A_f$ and pick $\gamma \in F^{-1}(\{B\}) \setminus A$. Since $a_\gamma \perp a_\alpha$ for every $\alpha \in A$,

$$a_\gamma \perp \sum_{f \in D} b_f,$$

thus, *a fortiori*, for every $i < n$,

$$a_\gamma \perp \sum_{f \in D,\, f(i)=1} b_f.$$

By Corollary 2.5, for every $i < n$,

$$3\mu_{\gamma,\beta_i} = \bigl(b_i^1/a_\gamma\bigr) = \left(b_i^1 - \sum_{f \in D,\, f(i)=1} b_f/a_\gamma\right) = (b_{\chi_i}/a_\gamma).$$

Therefore

$$c = \sum_{f\,:\, n \to 2} b_f \geqslant \sum_{i<n} b_{\chi_i} > \sum_{i<n} 2\mu_{\gamma,\beta_i} a_\gamma,$$

and, by (3.1),

$$\sum_{i<n} 2\mu_{\gamma,\beta_i} a_\gamma = 2\left(\sum_{i<n} \mu_{\gamma,\beta_i}\right) a_\gamma > 2(c/a_\gamma) a_\gamma.$$

Consequently $c > 2(c/a_\gamma)a_\gamma$, hence $(c/a_\gamma) \geqslant 2(c/a_\gamma)$, which is not the case. $\square$

Let L be the sublattice of $\mathcal{P}(\omega_1)$ consisting of all subsets x of ω_1 where

$$x = \bigcup_{i<n} [\alpha_i, \beta_i)$$

for some natural number n and $0 \leqslant \alpha_0 < \beta_0 < \cdots < \alpha_{n-1} < \beta_{n-1} \leqslant \omega_1$. Put

$$B = \bigl\{x \in L \mid \exists \alpha \in \omega_1\colon\ x \subseteq [0, \alpha)\bigr\},$$
$$U = \bigl\{y \in L \mid \exists \alpha \in \omega_1\colon\ y \supseteq [\alpha, \omega_1)\bigr\}.$$

Note that B is an ideal of the lattice L and U is the dual filter of the ideal B.

Let S be a 0-semilattice. Put $S^* = S\backslash\{0\}$, and set

$$T = \big(\{0\} \times B\big) \cup \big(S^* \times U\big) \subseteq S \times L. \tag{3.2}$$

Lemma 3.3. *If S is a distributive 0-semilattice, then the semilattice T defined by* (3.2) *is distributive.*

Proof. Let $x \leqslant y \vee z$ for some $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2) \in T$. We prove that there exist $y' \leqslant y$, $z' \leqslant z \in T$ with $x = y' \vee z'$.

In case $x_1 = 0$, we put $y_2' = x_2 \wedge y_2$, $z_2' = x_2 \wedge z_2$. Since $x_2 \in B$ and B is an ideal of L, $y_2', z_2' \in B$. Then $y' = (0, y_2')$, $z' = (0, z_2') \in T$ and $x = y' \vee z'$.

Let $x_1 > 0$. Since the semilattice S is distributive, there exist $y_1' \leqslant y_1$, $z_1' \leqslant z_1$ in S with $x_1 = y_1' \vee z_1'$. First assume that one of the elements y_1', z_1', say z_1', equals 0. Then $y_1' > 0$, and so $y_2 \in U$, hence there exists $\alpha < \omega_1$ such that $y_2 \supseteq [\alpha, \omega_1)$, whence $x_2 \leqslant y_2 \vee (z_2 \cap [0, \alpha))$. We put $y_2' = x_2 \wedge y_2$ and $z_2' = x_2 \wedge (z_2 \cap [0, \alpha))$. Then $y_2' \in U$, $z_2' \in B$, and $y' = (x_1', y_2')$, $z' = (0, z_2')$ are elements of T with $x = y' \vee z'$. If $0 < y_1', z_1'$, then $x_2, y_2, z_2 \in U$, hence $y_2' = x_2 \wedge y_2$, $z_2' = x_2 \wedge z_2 \in U$. By the definition, both $y' = (y_1', y_2')$, $z' = (z_1', z_2')$ are elements of T and $x = y' \vee z'$. $\square$

Lemma 3.4. *Let S be a distributive* 0*-semilattice which is not isomorphic to the maximal semilattice quotient of the positive cone of any weakly Archimedean dimension group. Let T be the semilattice defined by* (3.2)*. Then T is a distributive* 0*-semilattice which is not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group. Moreover* $\mathrm{card}(T) = \mathrm{card}(S)$.

Proof. By Lemma 3.1, $\mathrm{card}(S) \geqslant \aleph_1$, and so the equality $\mathrm{card}(T) = \mathrm{card}(S)$ is clear. The semilattice T is distributive by Lemma 3.3.

For $\alpha < \omega_1$ put

$$B_\alpha = \big\{x \in L \mid x \subseteq [0, \alpha)\big\}.$$

Given $\alpha < \omega_1$, set $I_\alpha = \{0\} \times B_\alpha$, and denote by I the ideal $\{0\} \times B$ of the semilattice T. For each $\alpha < \omega_1$ put $T_\alpha = T/\Theta[I_\alpha]$, and let $\varphi_{\alpha,\beta}$ $(\alpha < \beta < \omega_1)$ stand for the canonical projection $T_\alpha \to T_\beta$. Finally, denote by φ_α $(\alpha < \omega_1)$ the canonical projections $T_\alpha \to T/\Theta[I]$. Since I is a union of the increasing chain $\{I_\alpha \mid \alpha < \omega_1\}$ of ideals of the semilattice T, we see that

$$\big\langle T/\Theta[I], \varphi_\alpha\big\rangle_{\alpha<\omega_1} = \varinjlim \langle T_\alpha, \varphi_{\alpha,\beta}\rangle_{\alpha<\beta<\omega_1}.$$

Note that for every $(s, x), (t, y) \in T$, $(s, x) \equiv (t, y)(\Theta[I_\alpha])$ if and only if $s = t$ and $x \cap [\alpha, \omega_1) = y \cap [\alpha, \omega_1)$, and $(s, x) = (t, y)(\Theta[I])$ if and only if $s = t$. Hence the correspondence $(s, x)\Theta[I] \mapsto s$ defines an isomorphism from $T/\Theta[I]$ onto S.

To the contrary, suppose there exists a dimension group G and an isomorphism $\iota : \nabla(G^+) \to T$. Put

$$J_\alpha = \bigl\{a - b \mid a, b \in G^+ \text{ and } \iota\bigl([a]\bigr), \iota\bigl([b]\bigr) \in I_\alpha\bigr\} \quad (\alpha < \omega_1),$$
$$J = \bigl\{a - b \mid a, b \in G^+ \text{ and } \iota\bigl([a]\bigr), \iota\bigl([b]\bigr) \in I\bigr\}.$$

For each $\alpha < \omega_1$ denote by G_α the quotient G/J_α. Let $f_{\alpha,\beta}$ $(\alpha < \beta < \omega_1)$ stand for the canonical projection $G_\alpha \to G_\beta$, and denote by f_α $(a < \omega_1)$ the canonical projections $G_\alpha \to G/J$. The ideal J is the union of the increasing chain $\{J_\alpha \mid \alpha < \omega_1\}$ of ideals of the group G, hence

$$\langle G/J, f_\alpha\rangle_{\alpha<\omega_1} = \varinjlim \langle G_\alpha, f_{\alpha,\beta}\rangle_{\alpha<\beta<\omega_1}.$$

For every $\alpha < \omega_1$ denote by ι_α the isomorphism $\nabla(G_\alpha^+)$ onto T_α sending $[a + J_\alpha]$ to $\iota([a])\Theta[I_\alpha]$. It is easy to see that the diagram

$$\begin{tikzcd}
\nabla\bigl(G_\alpha^+\bigr) \arrow[r, "\iota_\alpha"] \arrow[d, "\nabla(f^+_{\alpha,\beta})"'] & T_\alpha \arrow[d, "\varphi_{\alpha,\beta}"] \\
\nabla\bigl(G_\beta^+\bigr) \arrow[r, "\iota_\beta"] & T_\beta
\end{tikzcd}$$

commutes for every $\alpha < \beta < \omega_1$. Since the functor ∇ preserves direct limits, there exists an isomorphism $\lambda : \nabla((G/J)^+) \to S \simeq T/\Theta[I]$ such that for every $a \in G^+$ and every $(s, x) \in T$, $\iota([a]) = (s, x)$ implies that $\lambda([a + J]) = s$.

Since, due to Lemma 3.2, S is not isomorphic to the maximal semilattice quotient of the positive cone of any weakly Archimedean dimension group, there exist $a', b' \in (G/J)^{++}$ such that $(a'/b') = \infty$. Put $s = \lambda([a'])$, $t = \lambda([b'])$, and pick $a, b \in G^+$ so that $\iota([a]) = (s, [0, \omega_1))$, and $\iota([b]) = (t, [0, \omega_1))$. Then $[a + J] = [a']$, $[b + J] = [b']$, hence, by Lemma 2.3(ii), $(a + J/b + J) = \infty$. By Lemma 2.7, there exists $\alpha < \omega_1$ with $(a + J_\alpha/b + J_\alpha) = \infty$. Note that

$$\iota_\alpha\bigl([a + J_\alpha]\bigr) = \bigl(s, [0, \omega_1)\bigr)\Theta[I_\alpha], \qquad \iota_\alpha\bigl([b + J_\alpha]\bigr) = \bigl(t, [0, \omega_1)\bigr)\Theta[I_\alpha].$$

Pick $c, d \in G^{++}$ with $\iota([c]) = (s, [\alpha + 1, \omega_1))$, and $\iota([d]) = (0, [0, \alpha + 1))$. Then $[a] = [c + d]$, hence $(c + d + J_\alpha/b + J_\alpha) = \infty$ by Lemma 2.3(ii). Further, $[b] > [d]$, which implies $(b + J_\alpha/d + J_\alpha) > 0$. Applying Lemma 2.6, we get that $(c + J_\alpha/b + J_\alpha) = \infty$, whence

$$\bigl(s, [\alpha + 1, \omega_1)\bigr)\Theta[I_\alpha] = \iota_\alpha\bigl([c + J_\alpha]\bigr) \geqslant \iota_\alpha\bigl([b + J_\alpha]\bigr) = \bigl(t, [0, \omega_1)\bigr)\Theta[I_\alpha]. \tag{3.3}$$

Observe that for every $(s, x), (t, y) \in T$, $(s, x)\Theta[I_\alpha] \geqslant (t, y)\Theta[I_\alpha]$ if and only if $s \geqslant t$ and $x \cap [\alpha, \omega_1) \supseteq y \cap [\alpha, \omega_1)$. Thus, according to (3.3), $[\alpha + 1, \omega_1) \supseteq [\alpha, \omega_1)$, which is a contradiction. □

Theorem 3.5. *There exists a distributive* 0*-semilattice* T *which is not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group. The size of the semilattice is* $\aleph_2$.

Proof. By Theorem 3.2 there exists a distributive 0-semilattice S, of size $\aleph_2$, which is not isomorphic to the maximal semilattice quotient of the positive cone of any weakly Archimedean dimension group. Apply Lemma 3.4. □

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